On the Kronecker Products

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Abstract:
In this project we concentrate on the Kronecker product and its properties and introduce the proofs of the main theorem related with.

Key words: the kronecker product, matrix calculus.

1. Introduction:
The kronecker product is very important in the areas of linear algebra and signal processing. It has a wide applications in system theory, matrix calculus, matrix equations, system identification and other special fields. The Kronecker product has an important role in the linear matrix equation theory. The solution of the Sylvester and the Sylvester-Like equation is a hotspot research area.

2. The Definition and examples:
2.1 Definition
Let $A \in R^{mn}$, $B \in R^{pq}$. Then the kronecker product of $A$ and $B$ is defined as the matrix

$$A \otimes B = \begin{bmatrix}
    a_{11}B & a_{12}B & \ldots & a_{1n}B \\
    a_{21}B & a_{22}B & \ldots & a_{2n}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}B & a_{m2}B & \ldots & a_{mn}B
\end{bmatrix} \in R^{mp \times nq}$$

Obviously, the same definition holds if $A$ and $B$ are complex-valued matrices. It is clear that the kronecker product of two diagonal matrices is a diagonal matrix and the kronecker of two upper (lower) triangular matrices is an upper (lower) triangular matrix.

In the next discussing, we denot to the identity matrix with order $m \times m$ by $I_m$. 

25
2.2 Example
1- For any $A \in \mathbb{R}^{p \times q}$, $I_m \otimes A = \text{diag}[A, A, \ldots, A]$ a block diagonal matrix with $n$ copies of $A$ along the diagonal.

2- Let $B$ an arbitrary $2 \times 2$ matrix. Then

$$B \times I_2 = \begin{bmatrix}
  b_{11} & 0 & b_{12} & 0 \\
  0 & b_{11} & 0 & b_{12} \\
  b_{21} & 0 & b_{22} & 0 \\
  0 & b_{21} & 0 & b_{22}
\end{bmatrix}$$

The extension to arbitrary $B$ and $I_n$ is obvious.

3- Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Then

$$x \otimes y \left[ x_1y^T, \ldots, x_my^T \right]^T$$

$$= \left[ x_1y_1, x_1y_n, x_2y_1, \ldots, x_my_1, \ldots, x_my_n \right]^T \in \mathbb{R}^{mn}$$

4- Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Then

$$x \otimes y^T \left[ x_1y, \ldots, x_my \right]^T$$

$$= \left[ x_1y_1, \ldots, x_1y_n, \vdots, \vdots, \vdots, \vdots, \vdots, x_my_1, \ldots, x_my_n \right]$$

$$= xy^T \in \mathbb{R}^{m \times n}$$

3. The basic properties of the Kronecker Product:

Let $A, B, C$ & $D$ matrices of the same order, then

1- $O \otimes A = A \otimes O = O$

2- $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$

3- $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

4- $A \otimes (B \otimes C) = (A \otimes B) \otimes C = A \otimes B \otimes C$

**Theorem 3.1.** Let $A \in R^{m \times n}, B \in R^{p \times q}, C \in R^{n \times p}$ and $D \in R^{s \times t}$.

Then, $(A \otimes B)(C \otimes D) = (AC \otimes BD) \in R^{mr \times st}$

**Proof:**
R.S. $(AC) \otimes (BD)$
\[
\begin{align*}
&= \left( \sum_{j=1}^{n} a_{ij} C_{j} \right) \otimes \left( \sum_{e=1}^{n} b_{ke} d_{en} \right) \\
&= \sum_{j=1}^{n} \left( a_{ij} B \right) C_{j} D \\
&= (A \otimes B)(C \otimes D) \\
&= (a_{ij} B) C_{j} D \\
&= \left( \sum_{j=1}^{n} a_{ij} C_{j} \right) BD \\
&= (AC) \otimes (BD)
\end{align*}
\]

3.2 Theorem: For all \( A \) and \( B \), \( (A \otimes B)^T = A^T \otimes B^T \)

The proof can be simply verified using the definitions of transpose and kronecker product.

3.3 Corollary: If \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times n} \) are symmetric, then \( A \otimes B \) is symmetric.

3.4 Theorem: If \( A \) and \( B \) are nonsingular. Then, \( (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \)

Proof: Using Theorem 3.1 simply note that
\[ (A \otimes B)(A^{-1} \otimes B^{-1}) = I \otimes I = I \]

3.5 Theorem: If \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times m} \) are normal

Then, \( A \otimes B \) is normal.

Proof: \( (A \otimes B)^T (A \otimes B) = (A^T \otimes B^T)(A \otimes B) \) by Theorem 3.2
\[
\begin{align*}
&= A^T A \times B^T B \\
&= AA^T \otimes BB^T \\
&= (A \times B)(A \times B)^T
\end{align*}
\]

since \( A \) \& \( B \) are normal

3.6 Definition:

Let \( A \) of order \( k \times \ell \). Then \( [A]^2 = A \otimes A \)

3.7 Theorem: Let \( A \) of order \( k \times \ell \). Then
\[
[A]^n = A \times A^{(n-1)}; \quad n = 2,3,4,\ldots
\]
On the Kronecker Products

Proof:
Using the mathematical proof:
1- The theorem is true when n=2 [from the definition]; \( A^{[2]} = A \otimes A \)
2- Suppose the theorem is true when \( n=m+1 \); \( A^{[m+1]} = A \times A^{[m]} \)
   But \( A \otimes A^{[m]} = A \otimes (A \otimes A^{[m-1]}) \)
   Thus, \( A^{[m+1]} = A \otimes A^{[m]} \)
   Which means that the theorem is true for all \( n \geq 2 \)
   We can generalize the above theorem for two matrices. The following theorem shows that.

3.8 Theorem: Let \( A = [a_{ij}]_{m \times n} \)
\( C = [c_{ne}]_{n \times r} \), then
\((AC)^{[n]} = A^{[n]} C^{[n]} \)

Proof: Using the mathematical proof
1- The theorem is true when \( n=1 \) where \( (AC)^{[1]} = AC = A^{[1]} C^{[1]} \), and also it is true when \( n=2 \) where
\((AC)^{[2]} = (AC) \otimes (AC) \)
\( = (A \otimes A)(C \otimes C) \)
\( = A^{[2]} C^{[2]} \)
2- We now suppose the theorem is true when \( n=k+1 \) and we have to proof that
\((AC)^{[k+1]} = A^{[k+1]} C^{[k+1]} \)
But
\((AC)^{[k+1]} = (AC) \otimes (AC)^{[k]} \) by Theorem 3.7
\( = (AC) \otimes A^{[k]} C^{[k]} \)
\( = (A \times A^{[k]})(C \otimes C^{[k]}) \)
\( = A^{[k+1]} C^{[k+1]} \)
This means that the theorem is true for all \( n \geq 2 \).

Some properties of the Vector Operator:
In this section we introduce a vector-valued operator.
Let \( A = [a_1, a_2, \ldots, a_n] \in F^{m \times n} \), where \( a_j = F^m \), \( j = 1, 2, \ldots, n \)
The vector \( \text{col}[A] \) is defined by
\[ \text{col}[A] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^{mn} \]

3.9 Theorem Let \( A \in F^{m \times n} \), \( B \in F^{n \times p} \) and \( c \in F^{p \times n} \), then

(1) \((I_p \otimes A)\text{col}[B] = \text{col}[AB]\)

(2) \((A \otimes I_p)\text{col}[C] = \text{col}[CA^T]\)

Proof. Let \((B)_j\) denote the \(i\)th column of matrix \(B\); we have

\[
(I_p \otimes A)\text{col}[B] = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}\begin{bmatrix} (B)_1 \\ (B)_2 \\ \vdots \\ (B)_p \end{bmatrix} = \text{col}[AB]
\]

Similarly, we have

\[
(A \otimes I_p)\text{col}[C] = \begin{bmatrix} a_{11}I_p & a_{12}I_p & \cdots & a_{1n}I_p \\ a_{21}I_p & a_{22}I_p & \cdots & a_{2n}I_p \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}I_p & a_{m2}I_p & \cdots & a_{mn}I_p \end{bmatrix}\begin{bmatrix} (C)_1 \\ (C)_2 \\ \vdots \\ (C)_n \end{bmatrix} = \begin{bmatrix} a_{11}(C)_1 + a_{12}(C)_2 + \cdots + a_{1n}(C)_n \\ a_{21}(C)_1 + a_{22}(C)_2 + \cdots + a_{2n}(C)_n \\ \vdots \\ a_{m1}(C)_1 + a_{m2}(C)_2 + \cdots + a_{mn}(C)_n \end{bmatrix}
\]
On the Kronecker Products

\[
\begin{bmatrix}
C(A^T)_1 \\
C(A^T)_2 \\
\vdots \\
C(A^T)_m
\end{bmatrix} = \begin{bmatrix}
(CA^T)_1 \\
(CA^T)_2 \\
\vdots \\
(CA^T)_m
\end{bmatrix} = \text{col}[CA^T]
\]

3.10 Theorem Let \( A \in F^{m \times n}, B \in F^{n \times p} \) and \( C \in F^{p \times q} \). Then \( \text{col}[ABC] = (C^T \otimes A)\text{col}[B] \)

Proof: According to Theorem 3.9 we have
\[
\text{col}[ABC] = \text{col}[(AB)C]
\]
\[
= (C^T \otimes I_m)\text{col}[AB]
\]
\[
= (C^T \otimes I_m)(I_p \otimes A)\text{col}[B]
\]
\[
= [(C^T \otimes I_m)(I_p \otimes A)]\text{col}[B]
\]
\[
= (C^T \otimes A)\text{col}[B]
\]

Theorem 3.10 plays an important role in solving the matrix equations[5], system identification[3].

Conclusion:
This study establishes some conclusions on the kronecker products. Proofs of main theorem on the kronecker product are given.
References